Second order equations

In general, we would like to solve higher order equations. That's generally impossible but there are some cases we can solve. We begin with linear equations of the second order. Everything we say is applicable to higher order but we focus on second order because those are the equations you will encounter.

Recall that -

$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = d(x)$ is the general form of a second order linear equation. If $d(x)=0$ the equation is homogeneous. Otherwise it is non-homogeneous.

In general, even this linear equation is impossible to solve for arbitrary functions $a(x), b(x), c(x)$, and $d(x)$ but there is one case we can always solve, the linear homogeneous equation with constant coefficients —

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \text{ where } a, b, c \text{ are constants.}$$

First you should recall the property of linear equations -

If y_1 and y_2 are each solutions of the equation, that is

$$a\frac{d^2y_1}{dx^2} + b\frac{dy_1}{dx} + cy_1 = 0 \quad \underline{\text{and}} \quad a\frac{d^2y_2}{dx^2} + b\frac{dy_2}{dx} + cy_2 = 0$$

then $y = c_1y_1 + c_2y_2$ is also a solution (where c_1 and c_2 are constants) because

$$\begin{aligned} a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy &= a\frac{d^2}{dx^2}(c_1y_1+c_2y_2) + b\frac{d}{dx}(c_1y_1+c_2y_2) + c(c_1y_1+c_2y_2) \\ &= c_1\left\{a\frac{d^2y_1}{dx^2} + b\frac{dy_1}{dx} + cy_1\right\} + c_2\left\{a\frac{d^2y_2}{dx^2} + b\frac{dy_2}{dx} + cy_2\right\} \\ &= c_1(0) + c_2(0) \\ &= 0 \end{aligned}$$

Then, in fact, if we can find 2 solutions then we have the general solution (with two arbitrary constants c_1 and c_2) by forming a linear combination $y = c_1y_1 + c_2y_2$ of the two solution that we found, y_1 and y_2.

There is one catch. In order to form a linear combination of two solutions those solutions must be linearly independent. In the second order case that means the two solutions can not be multiples of each other.

Ex! Suppose we find two solutions of a second order equation,

$y_1 = x$ and $y_2 = 5x$.

Then $y = c_1 y_1 + c_2 y_2$

$= c_1 x + c_2 (5x)$

$= (c_1 + 5c_2) x$

$= cx$ which is only one solution because $y_1 = x$ and $y_2 = 5x$ are not linearly independent.

The condition for independence is that if $c_1 y_1 + c_2 y_2 = 0$ then $c_1 = c_2 = 0$ if y_1 and y_2 are linearly independent. If not then $y_1 = \frac{-c_2}{c_1} y_1$ and the two solutions are multiples of each other.

This leads to a test. Suppose

$c_1 y_1 + c_2 y_2 = 0$

then $c_1 y_1' + c_2 y_2' = 0$ (by differentiating)

Solving the two equations in two unknowns gives

$c_1 = -\frac{c_2 y_2}{y_1}$

$-\frac{c_2 y_2}{y_1} y_1' + c_2 y_2' = 0$

or $c_2 (y_1 y_2' - y_2 y_1') = 0$

So if c_1 and c_2 are not zero then

$y_1 y_2' - y_2 y_1' = 0$ and the solutions are dependent.

This can be written as a determinant of a matrix. This determinant is called the Wronskian of y_1 and y_2 and denoted by $W(y_1, y_2)$.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

So if $W(y_1, y_2) = 0$ then the two solutions y_1 and y_2 are dependent and we need another solution. If $W(y_1, y_2) \neq 0$ then the two solutions y_1 and y_2 are linearly independent and $y = c_1 y_1 + c_2 y_2$.

ex: x and $5x$ are dependent because

$$W(x,5x) = \begin{vmatrix} x & 5x \\ 1 & 5 \end{vmatrix} = 5x - 5x = 0$$

but two solutions $y_1 = x$ and $y_2 = x^2$ are linearly independent because

$$W(x,x^2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2 \neq 0$$

So we are trying to solve the equation $ay'' + by' + cy = 0$ where a, b, c are constants. We need a function such that multiples of that function and its derivatives are zero. We happen to know that function, namely $y = e^{kx}$. (as an aside, you can prove that the only smooth function with this property is exponential).

So we assume $y = e^{kx}$ is a solution of $ay'' + by' + cy = 0$. Noting that $y' = ke^{kx}$ and $y'' = k^2 e^{kx}$ we plug it into the equation and see what happens —

$$ak^2 e^{kx} + bke^{kx} + ce^{kx} = 0$$

or $\quad ak^2 + bk + c = 0$

This quadratic in k is called the characteristic equation of the ODE. We can solve this equation for k —

$$k = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So we have two values for k, namely $k_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $k_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $y_1 = e^{k_1 x}$ and $y_2 = e^{k_2 x}$ should be two linearly independent solutions and $y = c_1 y_1 + c_2 y_2$ should be the general solution.

Actually, it's a bit more complicated. There are three cases to consider and these 3 cases depend on the value of the discriminant $b^2 - 4ac$. The three cases are $b^2 - 4ac > 0$, $b^2 - 4ac = 0$, and $b^2 - 4ac < 0$. We look at each individually.

Case 1: $b^2 - 4ac > 0$

If $b^2-4ac > 0$ then we have 2 distinct real roots, $k_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $k_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$ so $y_1 = e^{k_1 x}$ and $y_2 = e^{k_2 x}$ are two linearly independent solutions and $y = c_1 y_1 + c_2 y_2 = c_1 e^{k_1 x} + c_2 e^{k_2 x}$ is the general solution.

Ex: $y'' - y' - 2y = 0$.

We assume $y = e^{kx}$ is a solution. Substituting into the equation we find the characteristic equation

$$k^2 - k - 2 = 0$$

so $(k-2)(k+1) = 0$

$k = 2$ or $k = -1$

So $y_1 = e^{2x}$ and $y_2 = e^{-x}$ are two linearly independent solutions and

$$y = c_1 y_1 + c_2 y_2$$

$$\boxed{y = c_1 e^{2x} + c_2 e^{-x}}$$

is the general solution (you verify)

Ex: $y'' - 7y' = 0$

assume $y = e^{kx}$.

Then the characteristic equation is

$$k^2 - 7k = 0$$

$$k(k-7) = 0$$

$$k = 0, 7$$

and $y = c_1 e^{0x} + c_2 e^{7x}$

$$\boxed{y = c_1 + c_2 e^{7x}}$$

Ex: $\ddot{x} + 10\dot{x} + 21x = 0$

Assume $x = e^{kt}$

$$k^2 + 10k + 21 = 0$$

$$(k+3)(k+7) = 0$$

and $\boxed{x = c_1 e^{-3t} + c_2 e^{-7t}}$

Case 2 : $b^2 - 4ac < 0$

If $b^2 - 4ac < 0$, then $\sqrt{b^2-4ac} = i\sqrt{4ac-b^2}$ which is imaginary and $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$ are the complex roots of the characteristic equation.

If we let $\frac{-b \pm \sqrt{b^2-4ac}}{2a} = u \pm iv$ where $u = \frac{-b}{2a}$, $v = \frac{\sqrt{4ac-b^2}}{2a}$

Then the two solutions to the differential equation are

$y_1 = e^{(u+iv)x} = e^{ux}e^{ivx}$ and $y_2 = e^{(u-iv)x} = e^{u}e^{-ivx}$ and the general solution is $y = c_1y_1 + c_2y_2$

$$= c_1 e^{ux} e^{ivx} + c_2 e^{ux} e^{ivx}$$

We can rewrite this solution, Recall the Maclaurin series –

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

Then $e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \dots$

$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \frac{x^8}{8!} + \dots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right)$$

$$e^{ix} = \cos x + i\sin x$$

Then we also have

$$e^{-ix} = \cos(-x) + i\sin(-x)$$

But $\cos(-x) = \cos x$ because $\cos x$ is even

and $\sin(-x) = -\sin x$ because $\sin x$ is odd.

So $e^{-ix} = \cos x - i\sin x$

Combining these results our general solution becomes –

$$y = c_1 e^{ux} e^{ivx} + c_2 e^{ux} e^{-ivx}$$

$$= c_1 e^{ux}\left(\cos(vx) + i\sin(vx)\right) + c_2 e^{ux}\left(\cos(vx) - i\sin(vx)\right)$$

$$= e^{ux}\left\{(c_1+c_2)\cos(vx) + i(c_1-c_2)\sin(vx)\right\}$$

$$= e^{ux}\left\{c_1\cos(vx) + c_2\sin(vx)\right\}$$

Ex: $y''+4y'+5y=0$

assume $y=e^{kx}$

Then the characteristic equation is

$$k^2+4k+5=0$$

and $k=\dfrac{-4\pm\sqrt{16-20}}{2}$

$=-2\pm i$

So the 2 solutions are $y_1=e^{(-2+i)x}$ and $y_2=e^{(-2-i)x}$ and the general solution is

$$y=c_1e^{(-2+i)x}+c_2e^{(-2-i)x}$$

$$=c_1e^{-2x}e^{ix}+c_2e^{-2x}e^{-ix}$$

$$=c_1e^{-2x}\{\cos x+i\sin x\}+c_2e^{-2x}\{\cos x-i\sin x\}$$

$$=e^{-2x}\{(c_1+c_2)\cos x+i(c_1-c_2)\sin x\}$$

$$=\boxed{e^{-2x}\{c_1\cos x+c_2\sin x\}}$$

Ex: $y''-3y'+4y=0$

if $y=e^{kx}$

then $k^2-3k+4=0$

$k=\dfrac{3\pm\sqrt{9-16}}{2}$

$=\frac{3}{2}\pm i\frac{\sqrt{7}}{2}$

so $\boxed{y=e^{\frac{3}{2}x}\{c_1\cos(\tfrac{\sqrt{7}}{2}x)+c_2\sin(\tfrac{\sqrt{7}}{2}x)\}}$

Case 3: $b^2-4ac=0$

if $b^2-4ac=0$ then $k=\dfrac{-b\pm\sqrt{b^2-4c}}{2a}=-\dfrac{b}{2a}$ and $k=-\dfrac{b}{2a}$ is a repeated root, this is a problem because

$$y=c_1e^{-\frac{b}{2a}x}+c_2e^{-\frac{b}{2a}x}=(c_1+c_2)e^{-\frac{b}{2a}x}=ce^{-\frac{b}{2a}x}$$ and we only have one solution. ☹ We need a second, independent solution.

We can find a second solution using a method called "reduction of order". We assume that we know a solution y_1 and a second solution, y_2, is of the form $y_2 = y_1 v(x)$ where $v(x)$ is an unknown function of x. Let's plug it in to the original equation and see what happens.

First note that

$$y_2 = y_1 v$$
$$y_2' = y_1' v + y_1 v'$$
$$y_2'' = y_1'' v + 2y_1' v' + y_1 v''$$

Then if y_2 is a solution to our equation; $ay_2'' + by_2' + cy_2 = 0$.

So $$a\{y_1''v + 2y_1'v' + y_1 v''\} + b\{y_1'v + y_1 v'\} + cy_1 v = 0$$

$$\{ay_1'' + by_1' + cy_1\}v + 2ay_1'v' + ay_1 v'' + by_1 v' = 0$$

we know that the first term $ay_1'' + by_1' + cy_1 = 0$ because y_1 is a solution. So

$$2ay_1'v' + ay_1 v'' + by_1 v' = 0$$

But $y_1 = e^{-\frac{b}{2a}x}$, $y_1' = -\frac{b}{2a}e^{-\frac{b}{2a}x}$ so

$$2a\left(-\frac{b}{2a}e^{-\frac{b}{2a}x}\right)v' + ae^{-\frac{b}{2a}x}v'' + be^{-\frac{b}{2a}x}v' = 0$$

or $ae^{-\frac{b}{2a}x}v'' = 0$

so $v'' = 0$

$v' = c_1$

and $v = c_1 x + c_2$

So our second solution is $y_2 = y_1 v = (c_1 x + c_2)e^{-\frac{b}{2a}x} = xe^{-\frac{b}{2a}x}$ if we take $c_1 = 1$, $c_2 = 0$ because we only need one linearly independent solution

and the general solution is –

$$y = c_1 y_1 + c_2 y_2$$
$$y = c_1 e^{-\frac{b}{2a}x} + c_2 x e^{-\frac{b}{2a}x}$$

we can check independence –

$$W(y_1, y_2) = \begin{vmatrix} e^{-\frac{b}{2a}x} & xe^{-\frac{b}{2a}x} \\ -\frac{b}{2a}e^{-\frac{b}{2a}x} & e^{-\frac{b}{2a}x} - \frac{b}{2a}xe^{-\frac{b}{2a}x} \end{vmatrix} = e^{-\frac{b}{a}x} \neq 0$$

so y_1 and y_2 are indeed linearly independent!

Ex: $y'' - 8y' + 16y = 0$

assume $y = e^{kt}$. Then

$k^2 - 8k + 16 = 0$

$(k-4)(k-4) = 0$

and $k = 4$ is a repeated root

so $y_1 = e^{4x}$ is a solution and a second, linearly independent solution is $y_2 = xe^{4x}$ and the general solution is

$y = c_1 y_1 + c_2 y_2$

$$\boxed{y = c_1 e^{4x} + c_2 x e^{4x}}$$